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O(3) shift operators and the group SL(3, R)

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Abstract. A preliminary analysis of irreducible unitary representations of SL(3, R) is given using O(3) shift operator techniques similar to those used for treating SU(3) in an O(3) basis. A full analysis is given of the $\Delta l = 2$ representations, for which minimum l values of 0, $\frac{1}{2}$ and 1 are found, but not the $l_{\min} = \frac{3}{2}$ representations proposed by Biedenharn *et al.*

1. Introduction

In previous papers (Hughes 1973a, b, to be referred to as I, II) the O(3) content of irreducible unitary representations (IUR) of SU(3) was analysed, using two pairs of operators which shift the value of l (where $l(l+1)$ is the eigenvalue of the O(3) Casimir L^2) by ± 1 and ± 2. It was pointed out in II that very minor modification of these techniques would be needed in order to give a similar analysis of SL(3, R), which is the non-compact version of SU(3) containing O(3) as a maximal compact subgroup, as opposed to SU(2) which is a maximal compact subgroup of the non-compact version SU(2, 1). In this paper some preliminary results in this direction are given, but the analysis will not extend as far as considering l-degenerate states, whose treatment, difficult enough for SU(3), is even more complicated for SL(3, R) whose IUR are infinite dimensional.

The only IUR which will be completely analysed here are the harmonic oscillator-like $\Delta l = 2$ representations, for which no l degeneracy occurs. These have been considered by Biedenharn and collaborators and used to treat nuclear rotational motion (Biedenharn and Weaver 1972) and hadronic Regge sequences (Biedenharn *et al* 1972). In the latter paper they extend a proposal by Dothan, Gell-Mann and Ne'eman (Dothan *et al* 1965, Dothan and Ne'eman 1966) that SL(3, R) be considered as generating orbital excitations in hadronic states by suggesting that it can also function as the generator of combined orbital and spin excitations. Using a realization of the Lie algebra of SL(3, R) in terms of boson creation and annihilation operators, they derive four $\Delta l = 2$ IUR with minimum l values given by $l_{\min} = 0, \frac{1}{2}, 1$ and $\frac{3}{2}$; these they associate with, respectively, the π, ρ, N and Δ Regge trajectories. Whereas the first three IUR are fully confirmed by the techniques of this paper, they appear to exclude the $l_{\min} = \frac{3}{2}$ IUR.

The first step in a general analysis of IUR of SL(3, R) is to consider the restriction that the hermiticity conditions impose on the values of the invariants I_2 and I_3; this is found to restrict the possible IUR to one of two types. The next step is to obtain all possible minimum values of l (no maximum l value can be expected since the IUR must, with the exception of the identity representation, all be infinite dimensional); in doing this all the hermiticity requirements on the shift operators themselves must also be satisfied. The analysis up to this stage does not exclude the possibility of IUR with $l_{\min}$ having any non-negative half-integral value, although of course, except for the cases where $l_{\min} = 0, \frac{1}{2}$, or 1, none of them are of the $\Delta l = 2$ kind. Further analysis may be expected to impose further restrictions on the IUR which actually do occur, but this will not be pursued in the present paper since the problem of l degeneracy then arises.

Various products of the shift operators were written down explicitly in I (equations (35)–(44)) only for the case where m (the eigenvalue of the generator l_0 of O(3)) = 0. In order to deal with IUR containing half integral l, which clearly contain no $m = 0$ states, these expressions have to be modified. In fact their generalization to states of arbitrary m turns out to be extremely simple, and further, by a suitable renormalization of the shift operators, the m dependence of the products of these operators can be eliminated, although care must be taken when doing so. This is done in § 2, and the expressions given are valid for both SU(3) and SL(3, R). The transition from SU(3) to SL(3, R) is effected merely by changing the hermiticity requirements on the various operators, and these are listed also in § 2. The exact analysis of the $\Delta l = 2$ IUR, and the preliminary analysis of general IUR of SL(3, R), is given in § 3.

2. The group SL(3, R) and its shift operators

Like SU(3), the group SL(3, R) is generated by the O(3) generators l_i $(i = 0, \pm 1)$ and the operators q_μ $(\mu = 0, \pm 1, \pm 2)$ which transform under commutation with the l_i as a five-dimensional irreducible tensor representation of O(3). The commutation relations of the l_i, q_μ are exactly the same as for SU(3) and are given in (I, 4) and (I, 5). The difference between the two groups lies entirely in the hermiticity properties of the q_μ, namely $q_\mu^\dagger = q_{-\mu}$ for SU(3) and $q_\mu^\dagger = -q_{-\mu}$ for SL(3, R). This means that all operators constructed from the generators in I for SU(3) go over unchanged for SL(3, R), as do all their commutation relations and all equations connecting them which do not involve hermitian conjugation.

The eigenvalues of the group invariants I_2 and I_3 will still be denoted by

$$I_2 = \tfrac{1}{9}(p^2 + q^2 - pq + 3p) \tag{1}$$

and

$$I_3 = \tfrac{1}{162}(p - 2q)(2p + 3 - q)(p + q + 3). \tag{2}$$

Thus IUR of SL(3, R) will still be labelled by the pair (p, q) although, unlike for SU(3), they will not necessarily constitute a unique label. The other operators needed for our analysis are, apart from the O(3) invariant L^2, the third and fourth order O(3) scalar operators O_l^0 and Q_l^0, respectively, and the operators $O_l^{\pm j}$ $(j = 1, 2)$ which shift the l values of states upon which they act by $\pm j$. These operators are given in terms of the l_i, q_μ by equations (I, 6) to (I, 9) and (I, 14) to (I, 17). The states upon which these operators act will be chosen orthonormal and denoted by $|p, q, r; l, a_l, m\rangle$, where (p, q, r) labels the IUR and (l, a_l, m) the states within an IUR. r is an additional parameter needed if (p, q) do not alone uniquely determine the IUR, $l(l+1)$ and m are the eigenvalues of, respectively, L^2 and l_0, and a_l is an additional label used to distinguish states of degenerate l values which may be defined in terms of the eigenvalues of O_l^0 and Q_l^0 (see II).

The hermiticity properties of these operators differ from the case of SU(3), and depend on the orders to which they contain the q_μ. Thus, as for SU(3), I_2 and Q_l^0 are hermitian operators although I_2 is no longer positive definite, whereas I_3 and O_l^0 are anti-hermitian. Also, for $j = 1, 2$,

$$\langle p, q, r; l, a_l, m|(O_l^{+j})^\dagger|p, q, r; l+j, b_{l+j}, m\rangle$$
$$= \alpha_{j,l}\langle p, q, r; l, a_l, m|O_{l+j}^{-j}|p, q, r; l+j, b_{l+j}, m\rangle \tag{3}$$

where now

$$\alpha_{j,l} = -(2l+1)/(2l+2j+1). \tag{4}$$

As a consequence of this, equations (I, 28) to (I, 31) remain unchanged but in the first two of these the new values of $\alpha_{j,l}$ apply and imply that $O_{l\pm j}^{\mp j}\, O_l^{\pm j}$ are negative (instead of positive) definite hermitian operators, while (I, 33) and (I, 34) now contain a minus sign implying that $O_{l\pm 1}^{\mp 1}\, O_{l\pm 2}^{\mp 1}\, O_l^{\pm 2}$ and $O_{l\pm 2}^{\mp 2}\, O_{l\pm 1}^{\pm 1}\, O_l^{\pm 1}$ are anti-hermitian conjugates, as are $O_{l+1}^{-1}\, O_{l-1}^{+2}\, O_l^{-1}$ and $O_{l-1}^{+1}\, O_{l+1}^{-2}\, O_l^{+1}$.

This completes the summary of the modifications needed to the various SU(3) operators in order that they be applicable to SL(3, R). There is, however, a further minor sophistication which can be made and which applies equally to both groups. In I explicit expressions were given for those products of the shift operators which commute with L^2 in terms of the group invariants and O(3) scalar operators. They were, however, calculated only for the case when they act on states for which $m = 0$. This was good enough for SU(3), but clearly inadequate for investigating possible IUR of SL(3, R) containing half integral l, and consequently also m, values. As it turns out, the generalization to the case of arbitrary m is almost trivial, the only modifications to equations (I, 35), ..., (I, 44) for the product operators being to the denominators on their left-hand sides; these are that every factor of the type $(l+n)^2$ is replaced by $(l+m+n)(l-m+n)$. The right-hand sides of the equations are completely unchanged.

One may go further and define the operators $A_l^{\pm j}$ by

$$A_l^{+1} = [(l+m+1)(l-m+1)]^{-1/2} O_l^{+1} \tag{5}$$

$$A_l^{+2} = [(l+m+1)(l-m+1)(l+m+2)(l-m+2)]^{-1/2} O_l^{+2} \tag{6}$$

and

$$A_l^{-j} = A_{-(l+1)}^{+j}, \qquad j = 1, 2. \tag{7}$$

Note that the expression for A_l^{-j} in terms of O_l^{-j} contains $(l^2 - m^2)^{-1/2}$. This is exactly cancelled by the same factor in the matrix elements of O_l^{-j}, whose presence there guarantees that O_l^{-j} cannot act upon states with $l = |m|$ to give states with $l < |m|$. A_l^{-j} acts on states whose m dependence has been factored out and is therefore free of such constraints. In fact, when one then writes down for these operators the analogues of (I, 35), ..., (I, 44) one finds that the m dependent dominators are completely eliminated. Furthermore, the $A_l^{\pm j}$ satisfy precisely the same commutation relations with L^2, and the same hermiticity properties, whether for SU(3) or SL(3, R), as the corresponding $O_l^{\pm j}$. By using $A_l^{\pm j}$ instead of $O_l^{\pm j}$, one therefore completely eliminates all considerations of the O(3) subgroup from the analysis without any loss of generality, and this we do in the rest of this paper.

For ease of accessibility we list here the expressions for the most useful product operators; the others can easily be obtained from the corresponding formulae of I. They are

$$A_{l+1}^{-1} A_l^{+1} = -\tfrac{1}{9}(O_l^0)^2 - \tfrac{1}{3}(l+1)(l+3)Q_l^0 + 24l(l+1)^2(2l+3)I_2$$
$$-2l(l+1)^2(2l^3+8l^2+12l+27)$$

$$A_{l-1}^{+1}A_l^{-1} = -\tfrac{1}{9}(O_l^0)^2 - \tfrac{1}{3}l(l-2)Q_l^0 + 24l^2(l+1)(2l-1)I_2 - 2l^2(l+1)(2l^3-2l^2+2l-21) \tag{9}$$

$$A_{l+2}^{-2}A_l^{+2} = \tfrac{1}{9}(O_l^0)^2 + \tfrac{1}{3}(2l+3)(2l+5)Q_l^0 + 24(l+1)(l+4)(2l+3)^2 I_2 - 2l(l+1)(2l+3)(2l^3+25l^2+3l+3) \tag{10}$$

$$A_{l-2}^{+2}A_l^{-2} = \tfrac{1}{9}(O_l^0)^2 + \tfrac{1}{3}(2l-1)(2l-3)Q_l^0 + 24l(l-3)(2l-1)^2 I_2 - 2l(l+1)(2l-1)(2l^3-19l^2+9l+27) \tag{11}$$

$$\begin{aligned} 9A_{l-1}^{+1}A_{l-2}^{+1}A_l^{-2} &= -\tfrac{1}{3}(O_l^0)^3 - 3(l^2-3l+1)Q_l^0O_l^0 - \tfrac{1}{4}(2l-1)(l-3)(l-4)[Q_l^0, O_l^0] \\ &\quad + 432l(l-1)(2l-1)I_2O_l^0 + 1296\sqrt{6}l^2(l+1)(l-3)(2l-1)^2 I_3 \\ &\quad - 18l(l+1)(2l^4-4l^3+4l^2-26l+9)O_l^0. \end{aligned} \tag{12}$$

3. IUR of SL(3, R)

We start this section with a full analysis, similar to that given for SU(3) in II, of all the $\Delta l = 2$ IUR of SL(3, R), ie those for which l occurs in steps of two. These are the simplest IUR which can occur, since the $A_l^{\pm 1}$ have vanishing actions on all states, which may therefore be interconnected by only one pair of shift operators, the $A_l^{\pm 2}$, and consequently no possibility of l degeneracy arises. We shall suppose that all operators act to the right on states $|p, q, r; l, a_l, m\rangle$; however a_l is not needed since l is simple, l_0 commutes with all operators and their actions are all independent of m, and the values of p, q, r will always be clear, so we shall in fact omit the states from all equations.

The necessary conditions for the IUR (p, q, r) that $\Delta l = 2$ is that, for all l,

$$A_{l+1}^{-1}A_l^{+1} = A_{l-1}^{+1}A_l^{-1} = 0 \tag{13}$$

and

$$A_{l-1}^{+1}A_{l-2}^{+1}A_l^{-2} = 0 \tag{14}$$

since all these operators contain $A_l^{\pm 1}$.

Using (13) and equations (8), (9), we may solve for O_l^0 and Q_l^0 in terms of I_2 and l, obtaining

$$O_l^0 = \pm 3\sqrt{6}l(l+1)(4I_2+1)^{1/2} \tag{15}$$

and

$$Q_l^0 = 6l(l+1)(12I_2 - 2l^2 - 2l - 9). \tag{16}$$

Substituting for O_l^0 and Q_l^0 back into (10) and (11) then yields

$$A_{l+2}^{-2}A_l^{+2} = 24(l+1)^2(l+2)^2[9I_2 - l(l+3)] \tag{17}$$

$$A_{l-2}^{+2}A_l^{-2} = 24l^2(l-1)^2[9I_2 - (l+1)(l-2)]. \tag{18}$$

Further, using (14)–(16) in (12) gives

$$6I_3 = \pm I_2(4I_2+1)^{1/2} \tag{19}$$

where the sign in (19) corresponds to that in (15).

Squaring (19) and using the expressions (1) and (2) for I_2 and I_3 in terms of p and q, we obtain

$$q(p-q)(p+1)(p+3)(q+2)(p-q+2)=0. \tag{20}$$

There appear to be six possible cases, depending on which of the factors in (20) we take to be zero. In fact all cases give rise to either the same representations or their contragredients. If we take $q = 0$, we obtain

$$I_2 = \tfrac{1}{9}p(p+3), \qquad I_3 = \tfrac{1}{162}p(p+3)(2p+3) \tag{21}$$

where p is arbitrary. Had we taken $(p+1) = 0$, $(p+3) = 0$ or $(q+2) = 0$, we should have obtained the same set of values for I_2, I_3 but with p replaced by, respectively, $(q-1)$, q or $(p+1)$. $(p-q) = 0$ and $(p-q+2) = 0$ give rise to just the contragredients of these cases, ie $I_3 \to -I_3$. We may therefore assume, with no loss of generality, that $q = 0$.

Using the hermiticity conditions that I_2 be real and I_3 imaginary in (21) shows that $p = (-\frac{3}{2}+\mathrm{i}p_2)$, where p_2 is an arbitrary real number. This then gives

$$I_2 = -\tfrac{1}{4}-\tfrac{1}{9}p_2^2, \qquad I_3 = -\tfrac{1}{9}\mathrm{i}p_2(\tfrac{1}{4}+\tfrac{1}{9}p_2^2). \tag{22}$$

It is easy to check that for these values of I_2 the requirement that $A_{l\pm2}^{\mp2}A_l^{\pm2}$ be real and less than or equal to 0 for all l is satisfied.

The possible values of $l_{\min} \equiv \underline{l}$ are given by the equations $A_{\underline{l}-2}^{+2}A_{\underline{l}}^{-2} = 0$. Clearly, whatever the value of p_2, we can have $\underline{l} = 0$ or 1; however $\underline{l}$ can also arise from $9I_2-(\underline{l}+1)(\underline{l}-2) = 0$. With I_2 as in (22), this permits of real $\underline{l}$ only if $p_2 = 0$, in which case we obtain the repeated root $\underline{l} = \frac{1}{2}$. In this case we have $I_2 = -\frac{1}{4}$, $I_3 = 0$, so that this IUR is self-contragredient.

To summarize, there are an infinite number of $\Delta l = 2$ IUR with $\underline{l} = 0$ or 1, for which I_2 and I_3 are given by (22) and

$$O_l^0 = 2\sqrt{6}\mathrm{i}p_2l(l+1), \qquad Q_l^0 = -\tfrac{2}{3}[2p_2^2+3(l^2+l+6)] \tag{23}$$

$$A_{l+2}^{-2}A_l^{+2} = -24(l+1)^2(l+2)^2[\tfrac{9}{4}+p_2^2+l(l+3)] \tag{24}$$

$$A_{l-2}^{+2}A_l^{-2} = -24l^2(l-1)^2[\tfrac{9}{4}+p_2^2+(l+1)(l-2)] \tag{25}$$

where $-\infty < p_2 < \infty$. There is also one $\Delta l = 2$ IUR with $\underline{l} = \frac{1}{2}$ for which $I_2 = -\frac{1}{4}$, $I_3 = 0$ and

$$O_l^0 = 0, \qquad Q_l^0 = -2(l^2+l+6) \tag{26}$$

$$A_{l+2}^{-2}A_l^{+2} = -24(l+1)^2(l+\tfrac{3}{2})^2(l+2)^2 \tag{27}$$

$$A_{l-2}^{+2}A_l^{-2} = -24l^2(l-\tfrac{1}{2})^2(l-1)^2. \tag{28}$$

There is no $\Delta \underline{l} = 2$, $\underline{l} = \frac{3}{2}$ IUR, as was proposed by Biedenharn *et al* (1972).

We now turn to the general problem of classifying all IUR of SL(3, R). The first step is to use the hermiticity conditions for I_2, I_3 in (1) and (2) to find permissible values of $p \equiv p_1+\mathrm{i}p_2$ and $q \equiv q_1+\mathrm{i}q_2$. This is a straightforward process and we state only the results; these are that there are essentially two distinct types of cases:

(i) $q_2 = 0$, $q_1 = 2p_1+3$, so

$$I_2 = \tfrac{1}{9}[3(p_1+1)(p_1+3)-p_2^2] \tag{29}$$

$$I_3 = -\frac{\mathrm{i}p_2}{81}[9(p_1+2)^2+p_2^2] \tag{30}$$

where p_2 is arbitrary and $p_1 \geqslant -2$ (the latter restriction comes about from the observations that I_2, I_3 are unaltered by the replacement of p_1 by $-(p_1+4)$). The contragredient of any IUR (p_1, p_2) of this type is $(p_1, -p_2)$.
(ii) $p_1 = -2, q_1 = -1$, so

$$I_2 = -\tfrac{1}{9}(p_2^2+q_2^2-p_2q_2+3) \tag{31}$$

$$I_3 = -\tfrac{1}{162}\mathrm{i}(p_2-2q_2)(2p_2-q_2)(p_2+q_2) \tag{32}$$

where p_2 and q_2 are arbitrary real numbers satisfying $0 \leqslant q_2 \leqslant p_2$ (the latter restriction comes about from the observation that I_2, I_3 are unaltered if (p_2, q_2) is replaced by $(-p_2, q_2-p_2)$, $(q_2-p_2, -p_2)$, $(p_2-q_2, -q_2)$, $(-q_2, p_2-q_2)$, or (q_2, p_2)). The contragredient of any IUR (p_2, q_2) of this type is (p_2, p_2-q_2).

The most general intersection of these two types is when $p_1 = -2, q_1 = -1$, $q_2 = 0$, p_2 is arbitrary, and $I_2 = -\frac{1}{9}(p_2^2+3), I_3 = -\mathrm{i}p_2^3/8$. It should be noted that we are not claiming that IUR occur for all values of (p_1, p_2) and (p_2, q_2) above, only that they are not excluded by the hermiticity requirements on I_2, I_3 alone.

The next step is to classify all possible values for $l_{\min} \equiv \underline{l}$, and these are determined by the requirement that $\underline{l}$ be simple and $A_{\underline{l}}^{-j}|\underline{l}\rangle = 0$ for $j = 1, 2$. These imply that

$$A_{\underline{l}-1}^{+1}A_{\underline{l}}^{-1}|\underline{l}\rangle = A_{\underline{l}-2}^{+2}A_{\underline{l}}^{-2}|\underline{l}\rangle = 0. \tag{33}$$

Substituting these equations into (9) and (11) one finds that, providing $\underline{l} \neq 1$,

$$Q_{\underline{l}}^0 = -6\underline{l}[12(2\underline{l}-1)I_2-(\underline{l}+1)(2\underline{l}^2-10\underline{l}-9)] \tag{34}$$

$$(O_{\underline{l}}^0)^2 = 18\underline{l}^2(2\underline{l}-1)^2[12I_2-(\underline{l}+1)(\underline{l}-3)] \tag{35}$$

from which we see immediately that if $\underline{l} = 0$ then $Q_{\underline{l}}^0$ and $O_{\underline{l}}^0$ both vanish. If $\underline{l} = 1$, the two equations (33) become, on substituting in (9) and (11), identical, so that they do not on their own determine $Q_{\underline{l}}^0$ and $(O_{\underline{l}}^0)^2$ in terms of I_2. We shall deal with this case later and assume for the time being that $\underline{l} \neq 1$.

The conditions $A_{\underline{l}-1}^{+1}A_{\underline{l}-2}^{+1}A_{\underline{l}}^{-2} = 0$ leads, on substitution into equation (12), to the condition

$$18\sqrt{6}\underline{l}(2\underline{l}-1)I_3 = \pm\underline{l}(2\underline{l}-1)[24I_2-2(\underline{l}+1)(\underline{l}-3)]^{1/2}[\underline{l}(\underline{l}-2)-3I_2] \tag{36}$$

which is clearly satisfied if $\underline{l} = 0$ or $\frac{1}{2}$. We consider these two cases separately. Firstly, if $\underline{l} = 0, O_0^0$ and Q_0^0 both vanish and from equation (8) we see that $A_1^{-1}A_0^{+1} = 0$, so no $l = 1$ state occurs, whereas from (10) we see that $A_2^{-2}A_0^{+2} = 864I_2$, which satisfies the hermiticity requirements provided $I_2 < 0$. This is always true for type (ii) IUR, but for type (i) IUR only if $p_2^2 \geqslant 3(p_1+1)(p_1+3)$. Secondly, if $\underline{l} = \frac{1}{2}$, $O_{1/2}^0 = 0$ and $Q_{1/2}^0 = -243/4$, whereas $A_{3/2}^{-1}A_{1/2}^{+1} = 27(1+4I_2), A_{5/2}^{-2}A_{1/2}^{+2} = 54(48I_2-13)$. The hermiticity conditions $A_{3/2}^{-1}A_{1/2}^{+1} \leqslant 0$ and $A_{5/2}^{-2}A_{1/2}^{+2} \leqslant 0$ are again both satisfied for type (ii) IUR, but for type (i) IUR only if $p_2^2 \geqslant 3(p_1+\frac{3}{2})(p_1+\frac{5}{2})$.

To see if IUR with different values of $\underline{l}$ are permitted, divide (36) through by $\underline{l}(2\underline{l}-1)$, square and substitute for I_2 and I_3 in terms of p and q. This yields the equation

$$(\underline{l}+p+1)(\underline{l}-p-3)(\underline{l}+q)(\underline{l}-q-2)(\underline{l}+p-q)(\underline{l}-p+q-2) = 0 \tag{37}$$

to be satisfied by $\underline{l}$.

We treat the two types of IUR separately. Firstly, for type (ii) IUR (p_2, q_2), equation (37) becomes

$$|\underline{l}-1+\mathrm{i}p_2|^2|\underline{l}-1+\mathrm{i}q_2|^2|\underline{l}-1+\mathrm{i}(p_2-q_2)|^2 = 0 \tag{38}$$

so no value of $\underline{l}$ arises unless either $p_2 = 0, q_2 = 0$ or $p_2 = q_2$, and in these cases the only possible value of $\underline{l}$ is $\underline{l} = 1$; hence the only possible values for $\underline{l}$ for type (ii) IUR is $\underline{l} = 0$, $\frac{1}{2}$ or 1.

For type (i) IUR, (p_1, p_2), equation (37) becomes

$$|\underline{l}+p_1+1+\mathrm{i}p_2|^2|\underline{l}-p_1-3+\mathrm{i}p_2|^2(\underline{l}+2p_1+3)(\underline{l}-2p_1-5) = 0. \tag{39}$$

Now if $p_2 \neq 0$ we can have only $\underline{l} = -(2p_1+3)$ or $\underline{l} = (2p_1+5)$. The two cases are in fact identical since both give the same values for I_2, I_3, namely

$$I_2 = \tfrac{1}{9}[\tfrac{3}{4}(\underline{l}+1)(\underline{l}-3)-p_2^2], \qquad I_3 = -\frac{\mathrm{i}p_2}{81}[\tfrac{9}{4}(\underline{l}-1)^2+p_2^2] \tag{40}$$

so we need consider only $\underline{l} = (2p_1+5)$. We therefore have the possibility of IUR with $\underline{l} = \frac{3}{2}, 2, \frac{5}{2}, \ldots$ corresponding to the values $-\frac{7}{4}, -\frac{3}{2}, -\frac{5}{4}, \ldots$ of p_1. Using equations (8), (34), (35) and (40) we see that $A_{\underline{l}+1}^{-1}A_{\underline{l}}^{+1} \neq 0$, so none of these give IUR of the $\Delta l = 2$ type. Also from equation (35) we see that the hermiticity condition $(O_{\underline{l}}^0)^2 \leqslant 0$ implies that, if $\underline{l} \neq 0, \frac{1}{2}$ or 1, $12I_2-(\underline{l}+1)(\underline{l}-3) \leqslant 0$. For the above cases, for which I_2 is as given in (40), this expression $= -\frac{4}{3}p_2^2$, so these values of $\underline{l}$ are, so far as the analysis given up to now is concerned, perfectly consistent with hermiticity. In the particular case when $p_2 = 0$, $O_{\underline{l}}^0$ vanishes.

When $p_2 = 0$, it is also possible that $\underline{l}$ can arise from the vanishing of $(\underline{l}+p_1+1)$ or $(\underline{l}-p_1-3)$. Both cases give $I_2 = \frac{1}{3}\underline{l}(\underline{l}-2), I_3 = 0$, so we need consider only $\underline{l} = (p_1+3)$. For such cases, however, substituting for I_2 in equation (35) gives

$$(O_{\underline{l}}^0)^2 = 54\underline{l}^2(\underline{l}-1)^2(2\underline{l}-1)^2,$$

which is greater than or equal to 0 and hence violates hermiticity unless $\underline{l} = 0, \frac{1}{2}$ or 1. Hence no new values of $\underline{l}$ can arise in this way.

To summarize, apart from the $\Delta l = 2$ IUR, the analysis so far permits IUR with $\underline{l} = 0, \frac{1}{2}$ or 1 of both type (i) and (ii), and in addition it permits IUR of type (i), ie (p_1, p_2), with $\underline{l} = \frac{3}{2}, 2, \frac{5}{2}, \ldots$ corresponding to $p_1 = -\frac{7}{4}, -\frac{3}{2}, -\frac{5}{4}, \ldots$, and arbitrary p_2. If we were to continue the analysis further we should get involved in l degeneracy, so shall not do so in the present paper. However, we can use the well known fact that non-compact simple groups, like SL(3, R), possess, apart from the trivial representation, only infinite dimensional IUR, so we can use the criteria that $l_{\max} \equiv \bar{l}$ does not exist to obtain further restrictions on the values of p and q and possibly $\underline{l}$. In II we obtained for SU(3) the following equation to be satisfied by $\bar{l}$:

$$(\bar{l}-p)(\bar{l}+p+4)(\bar{l}-q+1)(\bar{l}+q+3)(\bar{l}-p+q+1)(\bar{l}+p-q+3) = 0. \tag{41}$$

It is not difficult to show that also for SL(3, R) if $\bar{l}$ exists it must be because of the vanishing of one of the factors in (41).

For type (i) representations (41) becomes

$$|\bar{l}-p_1+\mathrm{i}p_2|^2|\bar{l}+p_1+4+\mathrm{i}p_2|^2[\bar{l}-2(p_1+1)][\bar{l}+2(p_1+3)] = 0. \tag{42}$$

If $p_2 \neq 0$ $\bar{l}$ would exist if either $[\bar{l}-2(p_1+1)] = 0$ or $[\bar{l}+2(p_1+3)] = 0$; since $\bar{l}$ cannot exist we therefore have the restriction $p_1 \neq \frac{1}{2}(\underline{l}+k)$ where k is an integer greater than or equal to -2, and $p_1 \neq -\frac{1}{2}(\underline{l}+k)$ where k is an integer greater than or equal to -6 (this latter case could not occur anyway if $\underline{l}+k > 4$). Further, if $p_2 = 0$, we must exclude the possibilities that $(\bar{l}-p_1) = 0$ and $(\bar{l}+p_1+4) = 0$, and these imply that $p_1 \neq \underline{l}+j$ where j is an integer greater than or equal to -4. These are the further restrictions which apply to type (i) IUR with $l = 0, \frac{1}{2}$ or 1. For the type (i) IUR with $l = \frac{3}{2}, 2, \frac{5}{2}, \ldots$ a straightforward check shows that the above conditions imply no further restrictions on the values of p_1.

For type (ii) IUR equation (41) becomes

$$|l+2+\mathrm{i}p_2|^2|l+2+\mathrm{i}q_2|^2|l+2+\mathrm{i}(p_2-q_2)|^2 = 0 \tag{43}$$

none of these factors can vanish since l must be real and greater than or equal to 0 if it exists. Hence l cannot exist and no further restriction is obtained for type (ii) IUR.

Finally, in this paper we consider the values of O_l^0 and Q_l^0 for the case $l = 1$ since equations (34) and (35) are not valid in this case. When $l = 1$, the substitution $A_0^{+1}A_1^{-1} = 0$ in (8) and $A_{-1}^{+2}A_1^{-2} = 0$ in (9) both yield the same equation:

$$(O_1^0)^2 - 3Q_1^0 = 36(12I_2+19) \tag{44}$$

so do not determine both O_1^0 and Q_1^0 in terms of I_2. If, however, we substitute for Q_1^0 from (44) in equation (12) with $A_0^{+1}A_{-1}^{+1}A_1^{-2} = 0$, we get

$$O_1^0[(O_1^0)^2 - 72(9I_2+3)] = 7776\sqrt{6}I_3 \tag{45}$$

and substituting for I_2, I_3 in terms of (p, q) yields the following roots for the cubic equation (45) in O_1^0:

$$O_1^0 = 2\sqrt{6}(2p+3-q), \qquad O_1^0 = -2\sqrt{6}(p-2q), \qquad O_1^0 = -2\sqrt{6}(p+q+3) \tag{46}$$

For type (i) IUR the first value gives $O_1^0 = 4\mathrm{i}\sqrt{6}p_2$ which satisfies hermiticity, whereas the second and third values violate hermiticity unless $p_1 = -2$, in which case $O_1^0 = -2\sqrt{6}\mathrm{i}p_2$. For type (ii) IUR all three values satisfy hermiticity and yield the values $2\sqrt{6}\mathrm{i}(2p_2-q_2)$, $2\sqrt{6}\mathrm{i}(2q_2-p_2)$ and $-2\sqrt{6}\mathrm{i}(p_2+q_2)$ respectively. There would therefore appear to be three distinct IUR (p_2, q_2) with $l = 1$, all corresponding to different values of O_1^0, and just one such IUR (p_1, p_2) unless $p_1 = -2$ in which case there are two IUR with $l = 1$. Further analysis might, of course, impose further restrictions.

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